Strictly irreducible Markov operators and ergodicity properties of skew products

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joint work with Pablo Lummerzheim and Felix Pogorzelski

Random dynamical systems

Intuitively *random dynamical systems* consist of a set of transformations, which are chosen at random by a stationary and ergodic stochastic process.

Formally they are build of the following components.

① Shifts

- Consider a measurable space $(E, \mathscr{E})$.
- Let $(\Omega, \mathscr{C})$ denote the product space $(E^{\mathbb{N}_0}, \mathscr{E}^{\mathbb{N}_0})$.

  $\rightsquigarrow$ The shift S given by

  $$S(\omega_0\omega_1 \ldots \ldots) = \omega_1\omega_2 \ldots$$

  defines a measurable transformation on $(\Omega, \mathscr{C})$.
- Let ν be an S-invariant and ergodic P-measure on Ω

  $\rightsquigarrow$ MDS (Ω, ν, S) $\leftrightsquigarrow$ stochastic process

Random dynamical systems

② Families of transformations

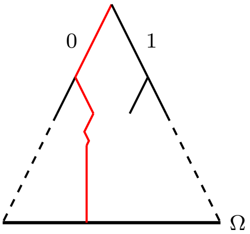
- Consider a probability space (X, μ).
- Let $(T_y)_{y \in E}$ be a measurable family of μ-preserving transformations of X.

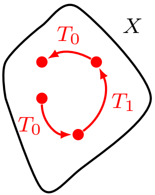
$\rightsquigarrow$ The skew product T on $\Omega \times X$ given by

$$T(\omega, x) := (S\omega, T_{\omega_0} x)$$

defines a $\nu \otimes \mu$-preserving transformation.

$\rightsquigarrow$ MDS $(\Omega \times X, \nu \otimes \mu, T)$ $\leftrightsquigarrow$ random dynamical system (RDS) $\leftrightsquigarrow$ "step skew product"





Random ergodic theorems

Assume that the family $(T_y)_{y\in E}$ is ergodic, i. e. any measurable set $A \subseteq X$ satisfying

$$T_y^{-1}(A) = A$$

for τ-almost all $y \in E$ has measure $\mu(A) \in \{0, 1\}$.

Fix $f \in L^1(X)$. Then for ν-almost every $\omega \in \Omega$ the random averages

$$\frac{1}{n}\sum_{i=0}^{n-1} f \circ T_{\omega_{i-1}} \circ \ldots T_{\omega_0}(x) = \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1} \otimes f \circ T^i(\omega, x)$$

converge for μ-almost all $x \in X$ by Birkhoff's ergodic theorem.

BUT: The limit function $\overline{f}$ may differ from the integral $\int f d\mu$!

Random ergodic theorems

Example

- Set $E := \{0,1\}$ and consider the sequences $\omega := 010101...$ and $\xi := 101010...$ $\rightsquigarrow$ The P-measure $\nu := {}^1\!/\!_2\, \delta_\omega + {}^1\!/\!_2\, \delta_\xi$ is S-invariant and ergodic.
- Set $X := \{x_1, x_2, x_3\}$, $\mu := ({}^1\!/\!_3, {}^1\!/\!_3, {}^1\!/\!_3)$. Let P be any permutation of X and set $T_0 := P$ and $T_1 := P^{-1}$.

Question: When does the ergodicity of the family $(T_y)_{y \in E}$ imply the ergodicity of the skew product T?

Theorem (Kakutani '51, Ryll-Nardzewski '55)

If the tranformations are chosen iid, i. e. ν is a product measure of the form $\tau^{\mathbb{N}_0}$ for some P-measure τ on $\mathscr{E}$, then the skew product T is ergodic if and only if the family $(T_y)_{y \in E}$ is ergodic.

What happens if we pass to Markov chains?

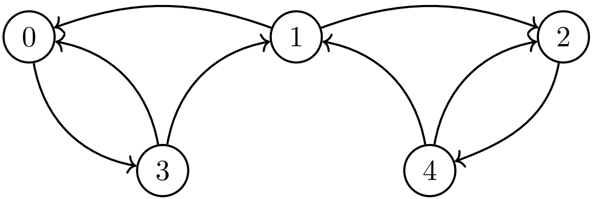
Finite state Markov chains

A Markov chain with finite state space $E = \{1, ..., k\}$ consists of

- a starting probability vector $\tau = (\tau_1, ..., \tau_k) \in \mathbb{R}^k$
- a row stochastic matrix $\Pi \in \mathbb{R}^{k \times k}$ consisting of transition probabilities π_{ij}

We assume that

- τ is a strictly positive fixed vector of $\Pi \rightsquigarrow$ stationarity
- Π is irreducible, i. e. for some $n \in \mathbb{N}$ the sum $\sum_{i=1}^{n} \Pi^i$ has only positive entries $\rightsquigarrow$ ergodicity



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By Kolmogorov's extension theorem there exists a unique probability measure ν on Ω satisfying

$$\nu(\{i_0\} \times \ldots \times \{i_{m-1}\} \times E \times \ldots) = \tau_{i_0} \pi_{i_0 i_1} \cdots \pi_{i_{m-2} i_{m-1}}$$

for all $i_0, \ldots, i_{m-1} \in E$ and $m \in \mathbb{N}$, which is S-invariant and ergodic.

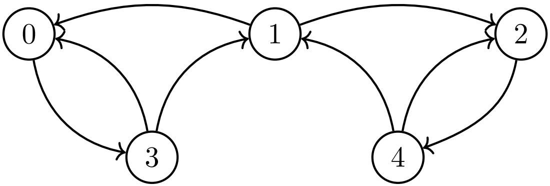
Strict irreducibility

Consider the relation $\sim$ on $E \times E$ arising between states $i, j \in E$ if either $i = j$ or $\pi_{ki} > 0$ and $\pi_{kj} > 0$ for third state $k \in E$.

In the present setting the following conditions are equivalent:

$$\begin{aligned} & \text{The graph } (E, \sim) \text{ is connected.} \\ \Leftrightarrow\quad & \text{The matrix } \Pi^T \Pi \text{ is irreducible.} \\ \Leftrightarrow\quad & \text{The matrix } \Pi\, \Pi^T \text{ is irreducible.} \end{aligned}$$

The matrix Π is called *strictly irreducible* if it satisfies one (and thus all) of the above conditions.



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Theorem (Bufetov '00)

Let Π be strictly irreducible. Then the skew product T is ergodic if and only if the family $\{T_1, ..., T_k\}$ is ergodic.

Question: Is this condition optimal? Can it be extended to general Markov chains?

General Markov chains

Let $(E, \mathscr{E})$ be a measurable space. A map $\pi\colon E \times \mathscr{E} \to [0,1]$ is called a *Markov kernel* if

- the component map $\pi(\cdot, B)$ is measurable for any $B \in \mathscr{E}$
- the component map $\pi(y, \cdot)$ is a probability measure for any $y \in E$ (which we denote by π_y in the following).

The product $\pi\kappa$ of two Markov kernels π and κ is given by

$$\pi\kappa(y, B) := \int_E \kappa(z, B)\ d\pi_y(z)$$

for $y \in E$ and $B \in \mathscr{E}$ and defines again a Markov kernel.

A probability measure τ on $\mathscr{E}$ is called π-*invariant* if for all $B \in \mathscr{E}$ we have

$$\tau(B) = \int_E \pi(y, B)\ d\tau(y).$$

General Markov chains

We call a Markov kernel π *irreducible* wrt a π-invariant measure τ if for all $B \in \mathscr{E}$ with $\tau(B) > 0$ there is for τ-almost every $y \in E$ some $n \in \mathbb{N}$ (which may depend on y) such that $\pi^n(y, B) > 0$.

A *general Markov chain* consists of

- measurable space $(E, \mathscr{E})$
- a probability measure τ on $(E, \mathscr{E})$
- a Markov kernel $\pi \colon E \times \mathscr{E} \to [0, 1]$

By Kolmogorov's extension theorem there exists a unique probability measure ν on Ω satisfying

$$\nu(B_0 \times ... \times B_{m-1} \times E \times ...) = \int_{B_0} \int_{B_1} ... \int_{B_{m-1}} d\pi_{y_{m-2}}(y_{m-1})...d\pi_{y_0}(y_1) d\tau(y_0)$$

for all $B_0, \dots, B_{m-1} \in \mathscr{E}$ and $m \in \mathbb{N}$.

Strict irreducibility of Markov kernels

We assume that τ is π-invariant and π is irreducible wrt τ.
$\rightsquigarrow$ The measure ν is S-invariant and ergodic.

We call a set $B \in \mathscr{E}$ *deterministic* if for τ-almost all $y \in E$ we have $\pi(y, B) \in \{0, 1\}$.

We shall say that the Markov kernel π is *strictly irreducible* wrt τ if every deterministic set $B \in \mathscr{E}$ has measure $\tau(B) \in \{0, 1\}$.

Remarks

- Generalization of the concept for finite state spaces. In this setting the minimal deterministic sets are given by the connected components of the graph $(E, \sim)$.
- Strict irreducibility implies irreducibility.

Excursion: Markov operators

A bounded linear operator $M\colon L^2(E,\tau) \to L^2(E,\tau)$ is called a *Markov operator* if

- $f \geq 0$ implies $Mf \geq 0$
- $M\mathbb{1} = \mathbb{1}$
- $\int Mf \, d\tau = \int f \, d\tau$ for all $f \in L^2(E,\tau)$.

A Markov operator M is called *irreducible* if for any $D \in \mathscr{E}$ with $M\mathbb{1}_D = \mathbb{1}_D$ we have $\tau(D) \in \{0,1\}$.

Remarks

- The class of Markov operators is closed under composition and taking adjoints.
- The Koopman operator associated to an MDS is always a Markov operator. It is irreducible if and only if the MDS is ergodic.

Markov kernels as Markov operators

A Markov kernel π with invariant probability measure τ gives rise to a Markov operator P defined by

$$Pf(y) := \int_E f \; d\pi_y$$

for $f \in L^2(E, \tau)$.

In the present setting we obtain the following equivalences:

- The irreducibility of π with respect to τ is equivalent to the irreducibility of P.
- PP^* is irreducible if and only if P^*P is irreducible and both is equivalent to the strict irreducibility of π with respect to τ.

Ergodicity of step skew products

Theorem (Lummerzheim-Pogorzelski-Z. ’23)

The following assertions are equivalent:

i) *The Markov kernel π is strictly irreducible wrt τ.*

ii) *Any step skew product T over S arising from a family $(T_y)_{y \in E}$ of mp transformations on some probability space is ergodic if and only if the family $(T_y)_{y \in E}$ is ergodic.*

Remarks

- generalizes Bufetov’s criterion from finite state spaces to arbitrary state space and shows that it is in fact a characterization.
- generalizes Kakutani’s theorem from Bernoulli processes to Markov chains.

Proof: Kowalski's theorem

Consider the Koopman operator $\widehat{T}$ on $L^2(\nu \otimes \mu)$ given by

$$\widehat{T}\varphi := \varphi \circ T.$$

The adjoint $\mathcal{L}_T$ of $\widehat{T}$ is called Perron-Frobenius operator.

Theorem (Kowalski '15): If $\varphi \in L^2(\nu \otimes \mu)$ is an eigenfunction of $\mathcal{L}_T$, then we have

$$\varphi(\omega, x) = \widehat{\varphi}(\omega_0, x) \quad \nu \otimes \mu\text{-almost surely}$$

for some $\widehat{\varphi} \in L^2(\tau \otimes \mu)$.

Observation: Every T-invariant function (i.e. any fixed function of $\widehat{T}$) is an eigenfunction of $\mathcal{L}_T$.

Corollary: Every T-invariant function $\varphi \in L^2(\nu \otimes \mu)$ satisfies

$$\varphi(\omega, x) := \widehat{\varphi}(\omega_0, x) \quad \nu \otimes \mu\text{-almost surely}$$

for some $\widehat{\varphi} \in L^2(\tau \otimes \mu)$.

Proof: $(i) \Rightarrow (ii)$

Let $D \subseteq \Omega \times X$ be a T-invariant set. We want to show that $D = \Omega \times A$ for some set $A \subseteq X$ invariant under $(T_y)_{y \in E}$.

Corollary $\rightsquigarrow$ There is a msb set $B \subseteq E \times X$ such that

$$\mathbb{1}_D(\omega, x) := \mathbb{1}_B(\omega_0, x) \quad \nu \otimes \mu\text{-a. s.}$$

For $x \in X$ set $B^x := \{y \in E \colon (y, x) \in B\}$, $g_x := \mathbb{1}_{B^x}$ and $h_x := \mathbb{1}_{E \setminus B^x}$.

Lemma: Let M be an irreducible Markov operator. Let $g, h \geq 0$ with $g + h = \mathbb{1}$ such that $\langle Mg, h \rangle = 0$. Then either $g = 0$ or $h = 0$.

Observation: The functions $\{g_x\}$ and $\{h_x\}$ satisfy

$$g_x(y) = P\Big\{g_{T_y x}\Big\}(y), \;\; h_x(y) = P\Big\{h_{T_y x}\Big\}(y)$$

for $\tau \otimes \mu$-almost all $(y, x) \in E \times X$.

$$
\begin{aligned}
0 &= \int_{E\times X} g_x(y)h_x(y)\ d\tau\otimes\mu(y,x) = \int_E\int_X P\Big\{g_{T_y x}\Big\}(y)P\Big\{h_{T_y x}\Big\}(y)\ d\mu(x)d\tau(y) \\
&= \int_E\int_X Pg_x(y)Ph_x(y)\ d\mu(x)d\tau(y) \\
&= \int_X\int_E Pg_x(y)Ph_x(y)\ d\tau(y)d\mu(x) = \int_X \langle Pg_x, Ph_x\rangle\ d\mu(x)
\end{aligned}
$$

This implies that for μ-almost all $x \in X$ we have

$$
\langle P^* Pg_x, h_x\rangle = \langle Pg_x, Ph_x\rangle = 0
$$

and thus, by the Lemma above, either

$$
\mathbb{1}_{B^x} = g_x = 0 \Rightarrow B_x = \varnothing
$$

or

$$
\mathbb{1}_{B^x} = \mathbb{1} - h_x = \mathbb{1} \Rightarrow B_x = E.
$$

This gives $B = E \times A$ for some msb $A \subseteq X$.

Easy: T-invariance of D implies A is invariant under $(T_y)_{y\in E}$.

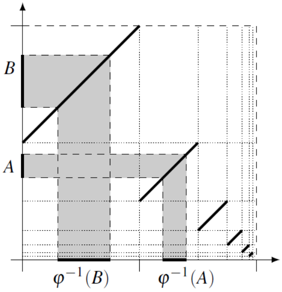
Proof: $(ii) \Rightarrow (i)$

Assume that π is not strictly irreducible wrt τ
$\rightsquigarrow$ deterministic set $B \subseteq E$ with $\tau(B) \in (0,1)$.

Consider the partition of E given by the sets

$$E_{B,B} := \{y \in B \colon \pi(y,B) = 1\},\ E_{B,B^c} := \{y \in B \colon \pi(y,B) = 0\}$$

$$E_{B^c,B^c} := \{y \in B^c \colon \pi(y,B^c) = 1\},\ E_{B^c,B} := \{y \in B^c \colon \pi(y,B^c) = 0\}.$$



Eisner/Farkas/Haase/Nagel:
Operator theoretic aspects
of ergodic theory, Springer 2015

Let d be the dyadic odometer on $[0,1)$ $\rightsquigarrow$ mp and ergodic wrt Lebesgue measure λ.

Set $I_1 := [0, {}^1\!/\!_2)$ and $I_2 := [{}^1\!/\!_2, 1)$ $\rightsquigarrow$ $d(I_1) = I_2$ and $d(I_2) = I_1$.

Define **ergodic** family $(T_y)_{y \in E}$ of λ-preserving transformations T_y on $[0,1)$ by

$$T_y := \begin{cases} d, & \text{if } y \in E_{B,B^c} \cup E_{B^c,B} \\ \mathrm{Id}, & \text{if } y \in E_{B,B} \cup E_{B^c,B^c} \end{cases}$$

Proof: $(ii) \Rightarrow (i)$

Claim: The corresponding skew product T is not ergodic.

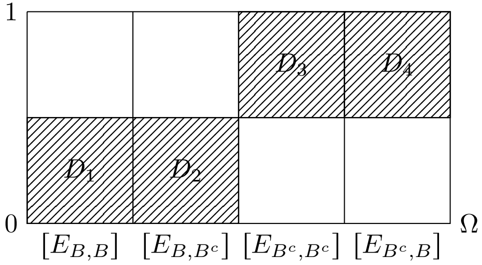
Consider the sets $D_1, \ldots, D_4 \subseteq \Omega \times [0,1)$ given by

$$D_1 := [E_{B,B}] \times I_1,\ D_2 := [E_{B,B^c}] \times I_1$$

$$D_3 := [E_{B^c,B^c}] \times I_2,\ D_4 := [E_{B^c,B}] \times I_2,$$

where $[M] := M \times E^{\mathbb{N}}$ for $M \subseteq E$, and set $D := D_1 \dot\cup \cdots \dot\cup D_4$.

$\rightsquigarrow$ D is a T-invariant set of measure $1/2$.



$$[E_{B,B}] \cup [E_{B,B^c}] = [B]$$

$$[E_{B^c,B^c}] \cup [E_{B^c,B}] = [B^c]$$

P. Lummerzheim, F. Pogorzelski and E. Zimmermann: *Strict irreducibility of Markov chains and ergodicity of skew products*, preprint, arxiv:2205.09847

Thank you!